

Low energy effective gravitational equations on a Gauss-Bonnet brane

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We present effective gravitational equations at low energies in a Z_2 -symmetric braneworld with the Gauss-Bonnet term. Our derivation is based on the geometrical projection approach, and we solve iteratively the bulk geometry using the gradient expansion scheme. Although the original field equations are quite complicated due to the presence of the Gauss-Bonnet term, our final result clearly has the form of the Einstein equations plus correction terms, which is simple enough to handle. As an application, we consider homogeneous and isotropic cosmology on the brane. We also comment on the holographic interpretation of bulk gravity in the Gauss-Bonnet braneworld.

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I. INTRODUCTION

Recent progress in string theory provides a revolutionary picture of the universe, in which our four-dimensional world is in fact a “braneworld” described by the motion of a thin wall embedded in a higher dimensional space-time. Various aspects on braneworld models have been addressed so far [1], and in a number of related publications the simplest Randall-Sundrum model [2] is employed for the analysis of gravity on the brane. In the Randall-Sundrum model, five-dimensional vacuum Einstein gravity with a negative cosmological constant is assumed. However, there will be higher derivative corrections in string gravity and hence it is natural to consider braneworld models with such corrections in the bulk action. While higher derivative corrections generally induce unwanted ill behavior, the special combination of curvature tensors called the *Gauss-Bonnet* term is known to avoid pathology: the Lagrangian is ghost-free, leading to well controlled field equations both at the classical and quantum levels [3]. In this paper we consider on the braneworld model described by the Einstein-Hilbert action plus the Gauss-Bonnet term.

In the context of the Gauss-Bonnet braneworld, cosmology [4, 5] and linearized gravity [6, 7] has been discussed so far in the literature. In the present paper we consider nonlinear gravity in the braneworld. In this direction, Maeda and Torii [8] have derived the effective gravitational equations on the brane using the geometrical projection approach. Though in the Randall-Sundrum model the geometrical projection is a highly successful approach to clarify the nature of brane gravity [9], Maeda and Torii’s effective equations in the Gauss-Bonnet braneworld are too complicated to be physically transparent. This is because the extrinsic curvature on the brane cannot be expressed solely by the energy-momentum tensor on it in a formal way. To cope with this problem, we shall resort to an approximation method. The approximation employed here is that the relevant length scales on the brane are much larger than the bulk curvature scale, focusing on gravitational physics at low energies.

The rest of the paper is organized as follows. In the next section we describe the model and present the basic equations following Ref. [8]. In Sec. III, we solve the bulk equation using the gradient expansion scheme and derive the effective equations on the Gauss-Bonnet brane. As the simplest example, we discuss homogeneous and isotropic cosmology on the brane in Sec. IV. Section V is devoted to summary and discussion.

II. BASIC EQUATIONS

The action we consider is given by

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} (\mathcal{R} - 2\Lambda + \alpha \mathcal{L}_{\text{GB}}) + \int d^4x \sqrt{-h} (Q + \mathcal{L}_m - \sigma), \quad (1)$$

where \mathcal{R} is the Ricci scalar of the five-dimensional metric g_{MN} and Λ is the cosmological constant in the bulk. $h_{\mu\nu}$ is the induced metric on the brane, σ is the brane tension, and \mathcal{L}_m is the Lagrangian for matter localized on the brane. The Gauss-Bonnet term \mathcal{L}_{GB} is defined as

$$\mathcal{L}_{\text{GB}} = \mathcal{R}^2 - 4\mathcal{R}_{MN}\mathcal{R}^{MN} + \mathcal{R}_{MNJK}\mathcal{R}^{MNJK}, \quad (2)$$

and the parameter α has dimension of (length)². The surface term Q is the Gibbons-Hawking-Myers term defined by [10]

$$Q = 2K + 4\alpha (J - 2G_\mu^\nu K_\nu^\mu), \quad (3)$$

where K_μ^ν is the extrinsic curvature of the brane and K is its trace, G_μ^ν is the four dimensional Einstein tensor with respect to the induced metric $h_{\mu\nu}$, and J is the trace of

$$J_\mu^\nu = \frac{1}{3} (2KK_\mu^\alpha K_\alpha^\nu + K_\alpha^\beta K_\beta^\alpha K_\mu^\nu - 2K_\mu^\alpha K_\alpha^\beta K_\beta^\nu - K^2 K_\mu^\nu). \quad (4)$$

In this paper we shall derive effective gravitational equations on a brane by solving the bulk geometry. For

this purpose, we use the (4+1)-decomposition of the metric and write the five-dimensional field equations in the form of the evolution equation along the extra dimension and the constraint equations. In deriving our basic equations, we will closely follow the geometrical projection approach by Maeda and Torii (henceforth MT) [8]. (See Appendix for an alternative approach based on the work by Deruelle and Katz [11].)

We write the bulk metric in the form

$$g_{MN}dx^M dx^N = dy^2 + q_{\mu\nu}(y, x)dx^\mu dx^\nu, \quad (5)$$

where y is the fifth coordinate. We may assume that the position of the brane is $y = 0$ without loss of generality, so that the induced metric on the brane is $h_{\mu\nu}(x) = q_{\mu\nu}(y = 0, x)$. We also assume a Z_2 -symmetry across the brane.

The traceless part of the evolution equation is given by

$$\begin{aligned} \mathcal{L}_n \tilde{K}_\mu^\nu &= -\frac{3}{2}E_\mu^\nu - \frac{1}{2}\tilde{M}_\mu^\nu \\ &\quad - \left(K_\mu^\alpha K_\alpha^\nu - \frac{1}{4}\delta_\mu^\nu K_\alpha^\beta K_\beta^\alpha \right), \end{aligned} \quad (6)$$

where \tilde{K}_μ^ν is the traceless part of the extrinsic curvature $K_\mu^\nu(y, x) = (1/2)q^{\alpha\nu}\partial_y q_{\mu\alpha}$,

$$E_{\mu\nu} := {}^{(5)}C_{\mu\alpha\nu\beta}n^\alpha n^\beta = {}^{(5)}C_{\mu y \nu y} \quad (7)$$

is the “electric” part of the five-dimensional Weyl tensor, and n^α is a hypersurface normal, $n^\alpha = (\partial_y)^\alpha$. M_μ^ν is the traceless part of $M_\mu^\nu := M_{\mu\alpha}{}^{\nu\alpha}$ where

$$M_{\mu\alpha\nu\beta} := R_{\mu\alpha\nu\beta}[q] - K_{\mu\nu}K_{\alpha\beta} + K_{\mu\beta}K_{\nu\alpha}. \quad (8)$$

and $R_{\mu\alpha\nu\beta}[q]$ is the four-dimensional Riemann tensor of the metric $q_{\mu\nu}(y, x)$.

The traceless part of the MT effective equations on the brane is arranged as

$$\begin{aligned} \frac{3}{2} \left(\tilde{M}_\mu^\nu + E_\mu^\nu \right) + \alpha \left[{}_{(a)}\bar{H}_\mu^\nu + {}_{(b)}\bar{H}_\mu^\nu + {}_{(c)}\bar{H}_\mu^\nu \right] \\ = -\frac{4\alpha}{3 + \alpha M} \tilde{M}_\mu^\nu \Lambda, \end{aligned} \quad (9)$$

where M is the trace of M_μ^ν . In the above we defined

$$\begin{aligned} {}_{(a)}\bar{H}_{\mu\nu} &:= 2 \left(L_{\mu\alpha\beta\gamma} L_\nu^{\alpha\beta\gamma} - \tilde{M}^{\alpha\beta} L_{\mu\alpha\nu\beta} - \tilde{M}_\mu^\alpha \tilde{M}_{\nu\alpha} \right) \\ &\quad - \frac{3 - \alpha M}{6(3 + \alpha M)} M \tilde{M}_{\mu\nu} + \frac{2\alpha}{3 + \alpha M} \tilde{M}_\alpha^\beta \tilde{M}_\beta^\alpha \tilde{M}_{\mu\nu} \\ &\quad - \frac{1}{2} q_{\mu\nu} \left(L_{\alpha\beta\rho\sigma} L^{\alpha\beta\rho\sigma} - \tilde{M}_{\alpha\beta} \tilde{M}^{\alpha\beta} \right), \end{aligned} \quad (10)$$

$$\begin{aligned} {}_{(b)}\bar{H}_{\mu\nu} &:= -3 \left(\tilde{M}_{\mu\alpha} E_\nu^\alpha + \tilde{M}_{\nu\alpha} E_\mu^\alpha + 2L_{\mu\alpha\nu\beta} E^{\alpha\beta} \right) \\ &\quad + \frac{3}{2} q_{\mu\nu} \tilde{M}_{\alpha\beta} E^{\alpha\beta} + \frac{1}{2} M E_{\mu\nu} \\ &\quad + \frac{6\alpha}{3 + \alpha M} \tilde{M}_{\alpha\beta} E^{\alpha\beta} \tilde{M}_{\mu\nu}, \end{aligned} \quad (11)$$

and

$$\begin{aligned} {}_{(c)}\bar{H}_{\mu\nu} &:= -4N_\mu N_\nu + 4N^\alpha (N_{\alpha\mu\nu} + N_{\alpha\nu\mu}) \\ &\quad + 2N_{\alpha\beta\mu} N^{\alpha\beta}_\nu + 4N_{\mu\alpha\beta} N_\nu^{\alpha\beta} \\ &\quad + 3q_{\mu\nu} \left(N_\alpha N^\alpha - \frac{1}{2} N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right) \\ &\quad + \frac{4\alpha}{3 + \alpha M} \left(N_\alpha N^\alpha - \frac{1}{2} N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right) \tilde{M}_{\mu\nu}, \end{aligned} \quad (12)$$

with

$$\begin{aligned} L_{\mu\nu\alpha\beta} &:= M_{\mu\nu\alpha\beta} + q_{\mu[\beta} \tilde{M}_{\alpha]\nu} + q_{\nu[\alpha} \tilde{M}_{\beta]\mu} \\ &\quad - \frac{1}{6} M q_{\mu[\alpha} q_{\beta]\nu}, \end{aligned} \quad (13)$$

$$N_{\mu\nu\alpha} := D_\mu K_{\nu\alpha} - D_\nu K_{\mu\alpha}, \quad (14)$$

$$N_\mu := q^{\alpha\beta} N_{\alpha\mu\beta} = D_\nu K_\mu^\nu - D_\mu K, \quad (15)$$

and D_μ is the covariant derivative with respect to the four-dimensional metric $q_{\mu\nu}$.

The trace part of the MT effective equations on the brane leads to the Hamiltonian constraint:

$$\frac{1}{2} M + \alpha \left[{}_{(a)}H + {}_{(b)}H + {}_{(c)}H \right] = \Lambda, \quad (16)$$

where $M := M_\mu^\mu = R[q] - K^2 + K_\mu^\nu K_\nu^\mu$. ${}_{(a)}H$, ${}_{(b)}H$, and ${}_{(c)}H$ are the trace of

$$\begin{aligned} {}_{(a)}H_{\mu\nu} &:= 2M_{\mu\alpha\beta\gamma} M_\nu^{\alpha\beta\gamma} - 6M^{\alpha\beta} M_{\mu\alpha\nu\beta} + 4M M_{\mu\nu} \\ &\quad - 8M_{\mu\alpha} M_\nu^\alpha - \frac{1}{8} q_{\mu\nu} (7M^2 - 24M_{\alpha\beta} M^{\alpha\beta} \\ &\quad + 3M_{\alpha\beta\rho\sigma} M^{\alpha\beta\rho\sigma}), \end{aligned} \quad (17)$$

$$\begin{aligned} {}_{(b)}H_{\mu\nu} &:= -6 \left(M_{\mu\alpha} E_\nu^\alpha + M_{\nu\alpha} E_\mu^\alpha + M_{\mu\alpha\nu\beta} E^{\alpha\beta} \right) \\ &\quad + \frac{9}{2} q_{\mu\nu} M_{\alpha\beta} E^{\alpha\beta} + 3M E_{\mu\nu}, \end{aligned} \quad (18)$$

and

$$\begin{aligned} {}_{(c)}H_{\mu\nu} &:= -4N_\mu N_\nu + 4N^\alpha (N_{\alpha\mu\nu} + N_{\alpha\nu\mu}) \\ &\quad + 2N_{\alpha\beta\mu} N^{\alpha\beta}_\nu + 4N_{\mu\alpha\beta} N_\nu^{\alpha\beta} \\ &\quad + 3q_{\mu\nu} \left(N_\alpha N^\alpha - \frac{1}{2} N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right). \end{aligned} \quad (19)$$

We require that an exact anti-de Sitter bulk with the curvature radius ℓ solves the five-dimensional field equations. Then the Hamiltonian constraint gives rise to the relation

$$\Lambda = -\frac{6}{\ell^2} \left(1 - \frac{\beta}{2} \right). \quad (20)$$

where we introduced a useful dimensionless quantity

$$\beta := \frac{4\alpha}{\ell^2}. \quad (21)$$

The Codacci equation, or the momentum constraint, is given by

$$\begin{aligned} D_\nu \left[K_\mu^\nu - \delta_\mu^\nu K \right. \\ \left. + 2\alpha \left(3J_\mu^\nu - J\delta_\mu^\nu - 2P_{\mu\alpha}{}^{\nu\beta} K_\beta^\alpha \right) \right] = 0, \end{aligned} \quad (22)$$

where

$$P_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + 2h_{\mu[\beta}R_{\alpha]\nu} + 2h_{\nu[\alpha}R_{\beta]\mu} + Rh_{\mu[\alpha}h_{\beta]\nu}. \quad (23)$$

The (generalized) Israel junction equations give the boundary condition at the brane as [12, 13]

$$K_{\mu}^{\nu} + \frac{\beta\ell^2}{3} \left(\frac{9}{2} J_{\mu}^{\nu} - J\delta_{\mu}^{\nu} - 3P_{\mu\alpha}^{\nu\beta} K_{\beta}^{\alpha} - G_{\alpha\beta} K^{\alpha\beta} \delta_{\mu}^{\nu} \right) = -\frac{\kappa^2}{6} \sigma \delta_{\mu}^{\nu} - \frac{\kappa^2}{2} \left(T_{\mu}^{\nu} - \frac{1}{3} \delta_{\mu}^{\nu} T \right), \quad (24)$$

where $T_{\mu\nu}$ is the energy-momentum tensor on the brane.

As a remark, we note that our basic equation (6) holds on any $y = \text{constant}$ hypersurfaces because the junction condition is not used in the derivation.

III. EFFECTIVE EQUATIONS ON THE BRANE

We will use the gradient expansion scheme [14, 15] in order to solve the bulk evolution equations, and then will derive the four-dimensional effective equations describing the brane geometry. The small expansion parameter here is the ratio of the bulk curvature scale to the brane intrinsic curvature scale,

$$\epsilon = \ell^2 |R|. \quad (25)$$

For any tensor A_{μ}^{ν} we expand it as

$$A_{\mu}^{\nu} = {}^{(0)}A_{\mu}^{\nu} + {}^{(1)}A_{\mu}^{\nu} + {}^{(2)}A_{\mu}^{\nu} + \dots, \quad (26)$$

where ${}^{(n)}A_{\mu}^{\nu} = \mathcal{O}(\epsilon^n)$.

A. Zeroth order equations

The zeroth order solution is given by

$${}^{(0)}K_{\mu}^{\nu} = -\frac{1}{\ell} \delta_{\mu}^{\nu}. \quad (27)$$

Since we have

$${}^{(0)}M_{\mu\nu}^{\alpha\beta} = \frac{1}{\ell^2} (-\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} + \delta_{\mu}^{\beta} \delta_{\nu}^{\alpha}), \quad (28)$$

$${}^{(0)}L_{\mu\nu}^{\alpha\beta} = 0, \quad (29)$$

one can confirm that Eq. (27) trivially solves the evolution equation, the traceless part of the MT effective equations, and the Codacci equation at zeroth order. It is easy to check that Eq. (27) satisfies the Hamiltonian constraint as well. The bulk metric is written in the form

$${}^{(0)}g_{MN} dx^M dx^N = dy^2 + a^2(y) h_{\mu\nu}(x) dx^{\mu} dx^{\nu}, \quad (30)$$

where the warp factor is given by

$$a = e^{-y/\ell}. \quad (31)$$

From the junction condition we obtain the following relation among the parameters:

$$\frac{1}{\ell} \left(1 - \frac{\beta}{3} \right) = \frac{1}{6} \kappa^2 \sigma. \quad (32)$$

B. First order equations

The zeroth order result in the previous subsection is rather trivial; going to higher order we will obtain the field equations that govern the brane geometry $h_{\mu\nu}(x)$.

First let us look at the trace part of the extrinsic curvature, which can be determined without solving the bulk geometry. It is easy to see that

$${}^{(1)}_{(a)}H = -\frac{2}{\ell^2} {}^{(1)}M, \quad {}^{(1)}_{(b)}H = {}^{(1)}_{(c)}H = 0, \quad (33)$$

and then the Hamiltonian constraint reduces to a simple form¹:

$$(1 - \beta) {}^{(1)}M = 0. \quad (34)$$

Thus we arrive at the same conclusion as the Randall-Sundrum braneworld:

$${}^{(1)}K = -\frac{\ell}{6} R[q] = -\frac{\ell}{6a^2} R[h]. \quad (35)$$

Let us move on to the investigation of the traceless part. It is straightforward to show that

$$\begin{aligned} {}^{(1)}_{(a)}\bar{H}_{\mu}^{\nu} &= -\frac{3 - \alpha {}^{(0)}M}{6(3 + \alpha {}^{(0)}M)} {}^{(0)}M {}^{(1)}\tilde{M}_{\mu}^{\nu} \\ &= \frac{2}{\ell^2} \frac{1 + \beta}{1 - \beta} {}^{(1)}\tilde{M}_{\mu}^{\nu}, \end{aligned} \quad (36)$$

$${}^{(1)}_{(b)}\bar{H}_{\mu}^{\nu} = \frac{1}{2} {}^{(0)}M {}^{(1)}E_{\mu}^{\nu} = -\frac{6}{\ell^2} {}^{(1)}E_{\mu}^{\nu} \quad (37)$$

$${}^{(1)}_{(c)}\bar{H}_{\mu}^{\nu} = 0, \quad (38)$$

and

$${}^{(1)}M_{\mu}^{\nu} = R_{\mu}^{\nu}[q] + \frac{2}{\ell} {}^{(1)}K_{\mu}^{\nu} + \frac{1}{\ell} {}^{(1)}K \delta_{\mu}^{\nu}. \quad (39)$$

Then the traceless part of the MT effective equations reads

$${}^{(1)}E_{\mu}^{\nu} = -{}^{(1)}\tilde{M}_{\mu}^{\nu}. \quad (40)$$

Substituting this into the traceless part of the evolution equation, we obtain

$$\mathcal{L}_n {}^{(1)}\tilde{K}_{\mu}^{\nu} = \tilde{R}_{\mu}^{\nu}[q] + \frac{4}{\ell} {}^{(1)}\tilde{K}_{\mu}^{\nu}, \quad (41)$$

¹ We assume $\beta \neq 1$ throughout the paper.

where $\tilde{R}_\mu^\nu[q]$ is the traceless part of the Ricci tensor $R_\mu^\nu[q] = R_\mu^\nu[h]/a^2(y)$. Note that Eq. (41) is again exactly the same as the first order evolution equation in the Randall-Sundrum model. The above equation can be integrated to give

$$^{(1)}\tilde{K}_\mu^\nu(y, x) = -\frac{\ell}{2a^2}\tilde{R}_\mu^\nu[h] + \frac{1}{a^4}^{(1)}\chi_\mu^\nu(x), \quad (42)$$

where $^{(1)}\chi_\mu^\nu(x)$ is an integration constant whose trace vanishes: $\chi_\mu^\mu = 0$. To avoid confusion, we explicitly show here that the Ricci tensor in the equation is constructed from the induced metric $h_{\mu\nu}(x)$. Combining Eq. (42) and the trace part (35), we obtain

$$^{(1)}K_\mu^\nu = -\frac{\ell}{2}\frac{1}{a^2}\left(R_\mu^\nu - \frac{1}{6}\delta_\mu^\nu R\right) + \frac{1}{a^4}^{(1)}\chi_\mu^\nu. \quad (43)$$

The extrinsic curvature (43) is related to the energy-momentum tensor on the brane via the junction condition. The junction condition at first order is

$$\begin{aligned} ^{(1)}K_\mu^\nu + \frac{\beta\ell^2}{3}\left(\frac{9}{2}^{(1)}J_\mu^\nu - ^{(1)}J\delta_\mu^\nu + \frac{3}{\ell}^{(1)}P_\mu^\nu - \frac{1}{\ell}^{(1)}P\delta_\mu^\nu\right) \\ = -\frac{\kappa^2}{2}\left(T_\mu^\nu - \frac{1}{3}\delta_\mu^\nu T\right). \end{aligned} \quad (44)$$

Substituting

$$^{(1)}J_\mu^\nu = -\frac{2}{3\ell^2}\left(^{(1)}K_\mu^\nu + 2^{(1)}K\delta_\mu^\nu\right) \quad (45)$$

and

$$^{(1)}P_{\mu\alpha}{}^{\nu\alpha} = -\left(R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R\right) \quad (46)$$

into the junction condition, we can rewrite Eq. (44) as

$$\begin{aligned} (1-\beta)^{(1)}K_\mu^\nu = -\frac{\kappa^2}{2}\left(T_\mu^\nu - \frac{1}{3}\delta_\mu^\nu T\right) \\ + \beta\ell\left(R_\mu^\nu - \frac{1}{6}\delta_\mu^\nu R\right). \end{aligned} \quad (47)$$

Then Eq. (47) together with Eq. (43) yields the gravitational equations on the brane at first order:

$$R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = \frac{\kappa^2}{\ell(1+\beta)}T_\mu^\nu + \frac{2}{\ell}\frac{1-\beta}{1+\beta}^{(1)}\chi_\mu^\nu. \quad (48)$$

We can see that for vanishing $^{(1)}\chi_\mu^\nu$ Einstein gravity is reproduced at this order.

From the coefficient of T_μ^ν we can read off the four-dimensional gravitational constant at low energies as

$$8\pi G = \frac{\kappa^2}{\ell(1+\beta)}. \quad (49)$$

In the linear perturbation analysis of Ref. [6] the scale-dependent gravitational coupling was obtained, and

Eq. (49) agrees with the result of [6] at long distances. Note, however, that the field equations (48) are nonlinear. This indicates that nonlinearity of gravity does not affect the gravitational coupling at low energies/long distances.

The integration constant $^{(1)}\chi_\mu^\nu$ is constrained from the Codacci equation:

$$\begin{aligned} D_\nu \left[^{(1)}K_\mu^\nu - \delta_\mu^\nu ^{(1)}K \right. \\ \left. + 2\alpha \left(3^{(1)}J_\mu^\nu - \delta_\mu^\nu ^{(1)}J + \frac{2}{\ell}^{(1)}P_\mu^\nu \right) \right] = 0, \end{aligned} \quad (50)$$

which implies that

$$(1-\beta)D_\nu \left[^{(1)}K_\mu^\nu - \delta_\mu^\nu ^{(1)}K \right] = 0. \quad (51)$$

Substituting Eq. (43) into the above we obtain

$$D_\nu ^{(1)}\chi_\mu^\nu = 0. \quad (52)$$

Namely, $^{(1)}\chi_\mu^\nu$ is transverse and traceless with respect to the brane geometry. This means that it corresponds to the dark radiation, and in fact we have $^{(1)}E_\mu^\nu = -(2/\ell)^{(1)}\chi_\mu^\nu$ on the brane. In what follows we set $^{(1)}\chi_\mu^\nu = 0$ for simplicity.

C. Second order equations

We now go on to the second order calculations in order to see corrections to four-dimensional general relativity (other than $^{(1)}\chi_\mu^\nu$).

First let us focus on the trace part. The Hamiltonian constraint implies that

$$\begin{aligned} (1-\beta)^{(2)}M = -\alpha \left[\frac{1}{6}^{(1)}M^2 - 2^{(1)}\tilde{M}_\mu^\nu{}^{(1)}\tilde{M}_\nu^\mu \right. \\ \left. + ^{(1)}L^{\alpha\beta}{}_{\rho\sigma}{}^{(1)}L_{\alpha\beta}{}^{\rho\sigma} \right]. \end{aligned} \quad (53)$$

From the definition of M and $L_{\alpha\beta}{}^{\rho\sigma}$ we have

$$^{(2)}M = \frac{6}{\ell}^{(2)}K - ^{(1)}K^2 + ^{(1)}K_\mu^\nu{}^{(1)}K_\nu^\mu \quad (54)$$

and

$$^{(1)}L_{\alpha\beta}{}^{\rho\sigma} = C_{\alpha\beta}{}^{\rho\sigma}, \quad (55)$$

where $C_{\alpha\beta\rho\sigma}$ is the four-dimensional Weyl tensor. Solving for $^{(2)}K$, we arrive at

$$\begin{aligned} ^{(2)}K = \frac{\ell}{6} \left[^{(1)}K^2 - ^{(1)}K_\mu^\nu{}^{(1)}K_\nu^\mu \right] \\ - \frac{\beta}{1-\beta}\frac{\ell^3}{24} \left[\mathcal{W} - 2^{(1)}\tilde{M}_\mu^\nu{}^{(1)}\tilde{M}_\nu^\mu + \frac{1}{6}^{(1)}M^2 \right], \end{aligned} \quad (56)$$

where \mathcal{W} is the trace of

$$\mathcal{W}_\mu{}^\nu := C_{\mu\alpha}{}^{\beta\gamma} C_{\beta\gamma}{}^{\nu\alpha}. \quad (57)$$

Recalling that $^{(1)}M = 0$ and $^{(1)}\tilde{M}_\mu{}^\nu = (2/\ell)^{(1)}\chi_\mu{}^\nu$ which is assumed to vanish, we obtain

$$^{(2)}K = -\frac{\ell^3}{24} \left(R_\mu{}^\nu R_\nu{}^\mu - \frac{1}{3} R^2 \right) - \frac{\beta}{1-\beta} \frac{\ell^3}{24} \mathcal{W}. \quad (58)$$

Next let us consider the traceless part. Again we need to solve the evolution equation to obtain the corresponding part of the brane extrinsic curvature.

A simple calculation leads to

$$\begin{aligned} ^{(2)}\bar{H}_\mu{}^\nu &= 2\tilde{\mathcal{W}}_\mu{}^\nu - 2^{(1)}\tilde{M}_\mu{}^\beta C_{\mu\beta}{}^{\nu\alpha} \\ &\quad - 2 \left[^{(1)}\tilde{M}_\mu{}^\alpha ^{(1)}\tilde{M}_\alpha{}^\nu - \frac{1}{4} \delta_\mu{}^\nu ^{(1)}\tilde{M}_\alpha{}^\beta ^{(1)}\tilde{M}_\beta{}^\alpha \right] \\ &\quad + \frac{2}{\ell^2} \frac{1+\beta}{1-\beta} ^{(2)}\tilde{M}_\mu{}^\nu, \end{aligned} \quad (59)$$

$$\begin{aligned} ^{(2)}\bar{H}_\mu{}^\nu &= 6 \left[^{(1)}\tilde{M}_\mu{}^\alpha ^{(1)}\tilde{M}_\alpha{}^\nu - \frac{1}{4} \delta_\mu{}^\nu ^{(1)}\tilde{M}_\alpha{}^\beta ^{(1)}\tilde{M}_\beta{}^\alpha \right] \\ &\quad + 6^{(1)}\tilde{M}_\alpha{}^\beta C_{\mu\beta}{}^{\nu\alpha} - \frac{6}{\ell^2} ^{(2)}E_\mu{}^\nu, \end{aligned} \quad (60)$$

$$^{(2)}\bar{H}_\mu{}^\nu = 0, \quad (61)$$

where we defined the traceless part $\tilde{\mathcal{W}}_\mu{}^\nu$ of $\mathcal{W}_\mu{}^\nu$. Then the traceless part of the MT effective equations reduces to

$$^{(2)}E_\mu{}^\nu = -^{(2)}\tilde{M}_\mu{}^\nu - \frac{\beta\ell^2}{3(1-\beta)} \frac{1}{a^4} \tilde{\mathcal{W}}_\mu{}^\nu. \quad (62)$$

Note here that we have $1/a^4(y)$ factored out so that $\tilde{\mathcal{W}}_\mu{}^\nu$ is computed from the induced metric $h_{\mu\nu}(x)$. We again omitted the dark radiation term $^{(1)}\chi_\mu{}^\nu \propto ^{(1)}\tilde{M}_\mu{}^\nu$.

From the definition of $\tilde{M}_\mu{}^\nu$ we have

$$\begin{aligned} ^{(2)}\tilde{M}_\mu{}^\nu &= ^{(2)}\tilde{R}_\mu{}^\nu + ^{(1)}K_\mu{}^\alpha ^{(1)}K_\alpha{}^\nu - \frac{1}{4} \delta_\mu{}^\nu ^{(1)}K_\alpha{}^\beta ^{(1)}K_\beta{}^\alpha \\ &\quad - ^{(1)}K^{(1)}\tilde{K}_\mu{}^\nu + \frac{2}{\ell} ^{(2)}\tilde{K}_\mu{}^\nu. \end{aligned} \quad (63)$$

Using Eqs. (62) and (63), we can write the evolution equation as

$$\begin{aligned} \mathcal{L}_n ^{(2)}\tilde{K}_\mu{}^\nu &= ^{(2)}\tilde{R}_\mu{}^\nu + \frac{4}{\ell} ^{(2)}\tilde{K}_\mu{}^\nu - ^{(1)}K^{(1)}\tilde{K}_\mu{}^\nu \\ &\quad + \frac{\beta\ell^2}{2(1-\beta)} \frac{1}{a^4} \tilde{\mathcal{W}}_\mu{}^\nu. \end{aligned} \quad (64)$$

The Ricci tensor $^{(2)}\tilde{R}_\mu{}^\nu$ in Eq. (64) can be calculated as follows. Integrating the first order result (43), we obtain the four-dimensional part of the bulk metric as

$$\begin{aligned} q_{\mu\nu}(y, x) &= a^2(y) \left[h_{\mu\nu}(x) + ^{(1)}g_{\mu\nu}(y, x) + \dots \right] \\ &= a^2 h_{\mu\nu} + \frac{\ell^2}{2} (a^2 - 1) \left[R_{\mu\nu} - \frac{1}{6} h_{\mu\nu} R \right] + \dots, \end{aligned} \quad (65)$$

where the boundary condition $^{(1)}g_{\mu\nu}(0, x) = 0$ is understood. Then the Ricci tensor is expressed as

$$\begin{aligned} ^{(2)}\tilde{R}_\mu{}^\nu(y, x) &= \frac{\ell^2}{2} (a^{-4} - a^{-2}) \left[R_\mu{}^\alpha R_\alpha{}^\nu - \frac{1}{6} R R_\mu{}^\nu \right. \\ &\quad \left. - \frac{1}{4} \delta_\mu{}^\nu \left(R_\alpha{}^\beta R_\beta{}^\alpha - \frac{1}{6} R^2 \right) \right. \\ &\quad \left. - \frac{1}{2} (D^\alpha D_\nu R_\alpha{}^\mu + D_\alpha D^\mu R_\nu{}^\alpha) \right. \\ &\quad \left. + \frac{1}{3} D_\mu D^\nu R + \frac{1}{2} D^2 R_\mu{}^\nu - \frac{1}{12} \delta_\mu{}^\nu D^2 R \right] \\ &=: \frac{\ell^2}{2} (a^{-4} - a^{-2}) \left[\mathcal{S}_\mu{}^\nu(x) + \frac{1}{6} R \tilde{R}_\mu{}^\nu \right], \end{aligned} \quad (66)$$

where $D^2 := D_\alpha D^\alpha$ and one should notice that $\mathcal{S}_\mu{}^\nu$ satisfies $\mathcal{S}_\mu{}^\mu = 0$ and

$$D_\nu \mathcal{S}_\mu{}^\nu = 0. \quad (67)$$

Using Eq. (66) and the first order result

$$^{(1)}K^{(1)}\tilde{K}_\mu{}^\nu = \frac{\ell^2}{12} \frac{1}{a^4} R[h] \tilde{R}_\mu{}^\nu[h], \quad (68)$$

we can integrate the evolution equation (64), yielding

$$\begin{aligned} ^{(2)}\tilde{K}_\mu{}^\nu &= \frac{\ell^2}{2} \left(\frac{y}{a^4} + \frac{\ell}{2} \frac{1}{a^2} \right) \mathcal{S}_\mu{}^\nu + \frac{1}{24} \frac{\ell^3}{a^2} R \tilde{R}_\mu{}^\nu \\ &\quad + \frac{\beta\ell^2}{2(1-\beta)} \frac{y}{a^4} \tilde{\mathcal{W}}_\mu{}^\nu + \frac{^{(2)}\chi_\mu{}^\nu(x)}{a^4}, \end{aligned} \quad (69)$$

where $^{(2)}\chi_\mu{}^\nu$ is an integration constant dependent only on the brane coordinates x^μ . To make the physical meaning of this integration constant clear, we define

$$\begin{aligned} ^{(2)}\bar{\chi}_\mu{}^\nu(x) &:= ^{(2)}\chi_\mu{}^\nu + \frac{\ell^3}{4} \mathcal{S}_\mu{}^\nu + \frac{\beta\ell^3}{6(1-\beta)} \tilde{\mathcal{W}}_\mu{}^\nu \\ &\quad + \frac{\ell^3}{8} \left[R_\mu{}^\alpha R_\alpha{}^\nu - \frac{1}{3} R R_\mu{}^\nu \right. \\ &\quad \left. - \frac{1}{4} \delta_\mu{}^\nu \left(R_\alpha{}^\beta R_\beta{}^\alpha - \frac{1}{3} R^2 \right) \right]. \end{aligned} \quad (70)$$

Now we can see that $^{(2)}E_\mu{}^\nu = -(2/\ell)^{(2)}\bar{\chi}_\mu{}^\nu$ on the brane. Thus we finally have the extrinsic curvature of the brane in terms of $\bar{\chi}_\mu{}^\nu$:

$$\begin{aligned} ^{(2)}K_\mu{}^\nu(y=0, x) &= \frac{\ell^3}{24} \left[-3R_\mu{}^\alpha R_\alpha{}^\nu + 2R R_\mu{}^\nu \right. \\ &\quad \left. - \frac{5}{12} \delta_\mu{}^\nu R^2 + \frac{1}{2} \delta_\mu{}^\nu R_\alpha{}^\beta R_\beta{}^\alpha \right] \\ &\quad - \frac{\beta\ell^3}{6(1-\beta)} \left[\mathcal{W}_\mu{}^\nu - \frac{3}{16} \delta_\mu{}^\nu \mathcal{W} \right] + ^{(2)}\bar{\chi}_\mu{}^\nu, \end{aligned} \quad (71)$$

which can be rearranged into

$$\begin{aligned} ^{(2)}K_\mu{}^\nu - \delta_\mu{}^\nu ^{(2)}K &= \frac{\ell^3}{2} \mathcal{P}_\mu{}^\nu \\ &\quad - \frac{\beta\ell^3}{6(1-\beta)} \left[\mathcal{W}_\mu{}^\nu - \frac{7}{16} \delta_\mu{}^\nu \mathcal{W} \right] + ^{(2)}\bar{\chi}_\mu{}^\nu, \end{aligned} \quad (72)$$

where

$$\mathcal{P}_\mu^\nu := \frac{1}{6}RR_\mu^\nu - \frac{1}{4}R_\mu^\alpha R_\alpha^\nu + \frac{1}{8}\delta_\mu^\nu R_\alpha^\beta R_\beta^\alpha - \frac{1}{16}\delta_\mu^\nu R^2. \quad (73)$$

The form of \mathcal{P}_μ^ν is suggestive. Rewriting this in terms of a new variable defined by $s_\mu^\nu := R_\mu^\nu - (1/2)\delta_\mu^\nu R$, we obtain the same expression as the well-known quadratic energy-momentum term which was first introduced in Ref. [9].

The Codacci equation at second order is summarized as

$$D_\nu \left[(1-\beta) \left({}^{(2)}K_\mu^\nu - \delta_\mu^\nu {}^{(2)}K \right) + \frac{\beta\ell^3}{2}\mathcal{Y}_\mu^\nu \right] = 0, \quad (74)$$

where

$$\begin{aligned} \mathcal{Y}_\mu^\nu &:= R_{\mu\alpha}{}^{\nu\beta}R_\beta^\alpha - RR_\mu^\nu + \frac{3}{2}R_\mu^\alpha R_\alpha^\nu \\ &\quad - \frac{3}{4}\delta_\mu^\nu R_\alpha^\beta R_\beta^\alpha + \frac{7}{24}\delta_\mu^\nu R^2 \\ &= C_{\mu\alpha}{}^{\nu\beta}R_\beta^\alpha - 2\mathcal{P}_\mu^\nu. \end{aligned} \quad (75)$$

The Codacci equation is formally integrated to give

$$\begin{aligned} (1-\beta) \left({}^{(2)}K_\mu^\nu - \delta_\mu^\nu {}^{(2)}K \right) + \frac{\beta\ell^3}{2}\mathcal{Y}_\mu^\nu \\ = \tau_\mu^\nu(x) + c_1\mathcal{S}_\mu^\nu + c_2\mathcal{Z}_\mu^\nu \end{aligned} \quad (76)$$

where τ_μ^ν is the part that cannot be expressed in terms of local quantities. As for the local part, \mathcal{Z}_μ^ν is the divergence free tensor defined by

$$\mathcal{Z}_\mu^\nu := RR_\mu^\nu - \frac{1}{4}\delta_\mu^\nu R^2 - D_\mu D^\nu R + \delta_\mu^\nu D^2 R, \quad (77)$$

and c_1 and c_2 are constant coefficients. Note that \mathcal{S}_μ^ν and \mathcal{Z}_μ^ν are the two linearly independent, divergence free combinations of curvature tensors which are of order ϵ^2 . This is because the variation of $\sqrt{-h}R^2$ and $\sqrt{-h}R_{\mu\nu}R^{\mu\nu}$ with respect to the metric gives \mathcal{Z}_μ^ν and $\mathcal{S}_\mu^\nu + \mathcal{Z}_\mu^\nu/3$, respectively. Due to the Gauss-Bonnet theorem in four dimensions, another curvature squared term $\sqrt{-h}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ does not give rise to a new independent local tensor quantity.

Substituting Eq. (72) into Eq. (76), we obtain

$$\begin{aligned} \tau_\mu^\nu(x) &= \frac{\beta\ell^3}{2}\mathcal{Y}_\mu^\nu - c_1\mathcal{S}_\mu^\nu - c_2\mathcal{Z}_\mu^\nu \\ &\quad + (1-\beta) \left[{}^{(2)}\bar{\chi}_\mu^\nu + \frac{\ell^3}{2}\mathcal{P}_\mu^\nu \right. \\ &\quad \left. - \frac{\beta\ell^3}{6(1-\beta)} \left(\mathcal{W}_\mu^\nu - \frac{7}{16}\delta_\mu^\nu \mathcal{W} \right) \right]. \end{aligned} \quad (78)$$

This equation relates the integration constant ${}^{(2)}\bar{\chi}_\mu^\nu$ to the nonlocal part τ_μ^ν and the free parameters c_1 and c_2 .

The traceless condition ${}^{(2)}\bar{\chi}_\mu^\mu = 0$ yields the constraint

$$\begin{aligned} \tau_\mu^\mu &= (1-3\beta)\frac{\ell^3}{8} \left[R_\mu^\nu R_\nu^\mu - \frac{1}{3}R^2 \right] \\ &\quad + \frac{\beta\ell^3}{8}\mathcal{W} - 3c_2D^2R. \end{aligned} \quad (79)$$

In the cases of $\beta = 0$ (the Randall-Sundrum model), τ_μ^μ represents the trace anomaly, which is expected from the AdS/CFT correspondence [16] in the braneworld model [17]. The nonlocal nature of $\tau_{\mu\nu}$ indicates that it corresponds to the energy-momentum of the holographic CFT on the brane. However, it is not trivial whether this is the case when $\beta \neq 0$, because the AdS/CFT correspondence in the presence of the Gauss-Bonnet term has not been addressed much yet [18].

Including the second order computation in the junction condition, we shall derive the effective equations on the brane with corrections to four-dimensional general relativity. The junction condition now becomes

$$\begin{aligned} (1-\beta) \left({}^{(1)}K_\mu^\nu - \delta_\mu^\nu {}^{(1)}K + {}^{(2)}K_\mu^\nu - \delta_\mu^\nu {}^{(2)}K \right) \\ = -\frac{\kappa^2}{2}T_\mu^\nu - \frac{\beta\ell^3}{2}\mathcal{Y}_\mu^\nu + \beta\ell G_\mu^\nu. \end{aligned} \quad (80)$$

Using the results obtained in the previous and present subsections, we have

$$G_\mu^\nu = \frac{\kappa^2}{\ell(1+\beta)}T_\mu^\nu + \frac{2}{\ell(1+\beta)}(\tau_\mu^\nu + c_1\mathcal{S}_\mu^\nu + c_2\mathcal{Z}_\mu^\nu), \quad (81)$$

or

$$\begin{aligned} G_\mu^\nu &= \frac{\kappa^2}{\ell(1+\beta)}T_\mu^\nu + \frac{2}{\ell} \frac{1-\beta}{1+\beta} {}^{(2)}\bar{\chi}_\mu^\nu \\ &\quad + \frac{(1-3\beta)\ell^2}{1+\beta}\mathcal{P}_\mu^\nu + \frac{\beta\ell^2}{1+\beta}C_{\mu\alpha}{}^{\nu\beta}R_\beta^\alpha \\ &\quad - \frac{\beta\ell^2}{3} \left[\mathcal{W}_\mu^\nu - \frac{7}{16}\delta_\mu^\nu \mathcal{W} \right]. \end{aligned} \quad (82)$$

Here the constants c_1 and c_2 are determined by the boundary condition other than that imposed at the brane. We stress that the above equations are correct even for nonlinear gravity, as long as they are applied to the low energy regime or at long distances.

IV. APPLICATION: COSMOLOGY ON THE BRANE

As an application, let us consider homogeneous and isotropic cosmology on the brane. We write the induced metric as

$$h_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (83)$$

where $a(t)$ is the scale factor. For this metric the Weyl tensor vanishes, $C_{\mu\nu\rho\sigma} = 0$, and hence we have $\mathcal{W}_\mu^\nu = 0$

and $\mathcal{Y}_\mu^\nu = -2\mathcal{P}_\mu^\nu$. Non-vanishing components of \mathcal{P}_μ^ν are given by

$$\mathcal{P}_t^t = -\frac{3}{4}H^4, \quad (84)$$

$$\mathcal{P}_i^j = -\left(\frac{3}{4}H^4 + H^2\dot{H}\right)\delta_i^j, \quad (85)$$

where $H := \dot{a}/a$ and a dot stands for a derivative with respect to t . Then a straightforward calculation shows that

$$D_\nu \mathcal{Y}_\mu^\nu = -2D_\nu \mathcal{P}_\mu^\nu = 0, \quad (86)$$

and thus the Codacci equation at second order implies

$$D_\nu {}^{(2)}\bar{\chi}_\mu^\nu = 0. \quad (87)$$

This means that ${}^{(2)}\bar{\chi}_\mu^\nu$ behaves as a conserved radiation like component on the cosmological brane (as is the case at first order). Assuming that this dark radiation term vanishes, we obtain the modified Friedmann equation from the (tt) component of the effective equations (82):

$$H^2 = \frac{\kappa^2}{3\ell(1+\beta)}\rho + \frac{(1-3\beta)\ell^2}{4(1+\beta)}H^4. \quad (88)$$

An exact form of the modified Friedmann equation in the Gauss-Bonnet braneworld was derived in Refs. [4, 8], which, omitting the dark radiation, is summarized as

$$\kappa^2(\rho + \sigma) = 2\sqrt{H^2 + \ell^{-2}}(3 - \beta + 2\beta\ell^2 H^2). \quad (89)$$

Expanding the right hand side of Eq. (89) for small $\epsilon = \ell^2 H^2$, we obtain the same expression as Eq. (88). Thus the validity of the gradient expansion is confirmed.

As long as one considers the homogeneous and isotropic universe on the brane, there is less advantage of our effective equations compared to [8]. However, it is sure that our procedure has a great advantage for general cases without symmetry. This is because we have the low energy effective theory at the nonlinear level.

V. SUMMARY AND DISCUSSION

In this paper we have obtained the effective gravitational equations at low energies in the ‘‘Gauss-Bonnet’’ braneworld. The derivation here is along the geometrical projection approach of Maeda and Torii [8]. At low energies we can solve iteratively the evolution equation in the bulk by expanding the relevant equations in terms of $\epsilon := (\ell/L)^2 \ll 1$, where ℓ is the bulk curvature scale and L is the brane intrinsic curvature scale. Up to second order in the gradient expansion, nonlinear gravity on the Gauss-Bonnet brane is described by Eq. (81) or Eq. (82), which is the main result of the present paper. Although the Gauss-Bonnet term makes the original governing equations complicated and far from transparent, our effective equations have the form of the Einstein

equations with correction terms, and are simple enough to handle. We can repeat the same procedure to include higher order effects.

In the low energy effective equations (81) we have the nonlocal tensor $\tau_{\mu\nu}$ and two free parameters corresponding to the bulk degrees of freedom. In the Randall-Sundrum model ($\beta = 0$), we can determine that part to close the equations via the AdS/CFT correspondence once we specify a CFT on the brane [14]. However, such a holographic interpretation of the bulk geometry is obscure in the presence of the Gauss-Bonnet term, and this point is an open question for further study. Not relying on the holographic argument, we can instead determine the integration constant $\chi_{\mu\nu}$ by imposing another boundary condition, for example, at a second brane introduced away from the first brane.

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APPENDIX A: COMMENT ON AN ALTERNATIVE FORMULATION

The main text is based on the Gauss-Bonnet brane equations derived by Maeda and Torii [8] using a $(4+1)$ decomposition first developed by Shiromizu *et al.* [9] to derive the Einstein brane equations. In this appendix we sketch a similar formulation of Gauss-Bonnet brane equations developed in Ref. [11] in the case of Einstein gravity, with an emphasis on the structure of the equations.

Consider a five dimensional spacetime in gaussian normal coordinates $x^A = (x^\rho, y)$ with line element

$$ds^2 = dy^2 + q_{\mu\nu}(x^\rho, y)dx^\mu dx^\nu \quad (A1)$$

and expand the metric coefficients $q_{\mu\nu}(x^\rho, y)$ near the surface $y = 0$ as

$$\begin{aligned} q_{\mu\nu}(x^\rho, y) &= g_{\mu\nu}(x^\rho) + k_{\mu\nu}(x^\rho)y + \frac{1}{2}l_{\mu\nu}(x^\rho)y^2 \\ &\quad + \mathcal{O}(y^3). \end{aligned} \quad (A2)$$

At lowest order in y the Riemann tensor of the met-

ric (A1) is

$$\begin{aligned}\mathcal{R}_{y\mu y\nu} &= -\frac{1}{2}l_{\mu\nu} + \frac{1}{4}k_{\rho\mu}k_{\nu}^{\rho}, \\ \mathcal{R}_{y\mu\nu\rho} &= -\frac{1}{2}D_{\nu}k_{\mu\rho} + \frac{1}{2}D_{\rho}k_{\mu\nu}, \\ \mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + \frac{1}{4}(k_{\mu\sigma}k_{\nu\rho} - k_{\mu\rho}k_{\nu\sigma}),\end{aligned}\quad (\text{A3})$$

where D_{μ} and $R_{\mu\nu\rho\sigma}$ are the covariant derivative and Riemann tensor of the metric $g_{\mu\nu}(x^{\rho})$.

The Einstein-Gauss-Bonnet field equations are

$$\Lambda\delta_B^A + \mathcal{G}_B^A + \alpha\mathcal{H}_B^A = 0 \quad (\text{A4})$$

where $\mathcal{G}_B^A := \mathcal{R}_B^A - (1/2)\delta_B^A\mathcal{R}$ is the five-dimensional Einstein tensor and where the Gauss-Bonnet tensor \mathcal{H}_B^A is defined as

$$\begin{aligned}\mathcal{H}_B^A &:= 2[\mathcal{R}^{ALMN}\mathcal{R}_{BLMN} - 2\mathcal{R}^{LM}\mathcal{R}^A{}_{LBM} \\ &\quad - 2\mathcal{R}^{AL}\mathcal{R}_{BL} + \mathcal{R}\mathcal{R}_B^A] - \frac{1}{2}\delta_B^A\mathcal{L}_{\text{GB}}.\end{aligned}\quad (\text{A5})$$

Expanding the Einstein tensor at lowest order in y is a straightforward calculation which yields

$$\mathcal{G}_{yy} = -\frac{1}{8}(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta})k^{\mu\nu}k^{\alpha\beta} - \frac{1}{2}R, \quad (\text{A6})$$

$$\begin{aligned}\mathcal{G}_{y\mu} &= \frac{1}{2}(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta})D^{\nu}k^{\alpha\beta} \\ &= \frac{1}{2}D_{\alpha}(k_{\mu}^{\alpha} - \delta_{\mu}^{\alpha}k),\end{aligned}\quad (\text{A7})$$

$$\begin{aligned}\mathcal{G}_{\mu\nu} &= -\frac{1}{2}(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta})l^{\alpha\beta} + \frac{1}{2}k_{\mu}^{\alpha}k_{\alpha\nu} - \frac{1}{4}kk_{\mu\nu} \\ &\quad + \frac{1}{8}g_{\mu\nu}(k^2 - 3k_{\sigma}^{\rho}k_{\rho}^{\sigma}) + G_{\mu\nu}.\end{aligned}\quad (\text{A8})$$

Expanding the Gauss-Bonnet tensor is only slightly more involved and yields

$$\begin{aligned}\mathcal{H}_{yy} &= -\frac{1}{16}(N_{\mu\alpha\nu\beta} - 4P_{\mu\alpha\nu\beta})k^{\mu\nu}k^{\alpha\beta} \\ &\quad - \frac{1}{2}(R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)\end{aligned}\quad (\text{A9})$$

$$\begin{aligned}\mathcal{H}_{y\mu} &= \frac{1}{2}N_{\mu\alpha\nu\beta}D^{\nu}k^{\alpha\beta} \\ &= \frac{1}{2}D_{\alpha}\left(\frac{3}{2}j_{\mu}^{\alpha} - \frac{1}{2}\delta_{\mu}^{\alpha}j - 4P_{\beta\mu\gamma}^{\alpha}k^{\beta\gamma}\right),\end{aligned}\quad (\text{A10})$$

$$\mathcal{H}_{\mu\nu} = -\frac{1}{2}N_{\mu\alpha\nu\beta}l^{\alpha\beta} + 2\overline{\mathcal{M}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{\mathcal{L}}, \quad (\text{A11})$$

where $P^{\mu\nu\rho\sigma}$ is defined in Eq. (23) in the main text, and the fully developed expressions of the other tensors are

$$j^{\mu\nu} := -\frac{2}{3}k^{\mu\rho}k_{\rho\sigma}k^{\sigma\nu} + \frac{2}{3}k^{\mu\rho}k_{\rho}^{\nu} + \frac{1}{3}k^{\mu\nu}(k_{\sigma}^{\rho}k_{\rho}^{\sigma} - k^2), \quad (\text{A12})$$

$$\begin{aligned}N^{\mu\alpha\nu\beta} &:= -(k^{\mu\beta}k^{\alpha\nu} - k^{\mu\nu}k^{\alpha\beta}) + \frac{1}{2}(g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta})(k_{\sigma}^{\rho}k_{\rho}^{\sigma} - k^2) - k^{\mu\rho}k_{\rho}^{\beta}g^{\alpha\nu} + k^{\alpha\rho}k_{\rho}^{\beta}g^{\mu\nu} \\ &\quad - k^{\alpha\rho}k_{\rho}^{\nu}g^{\mu\beta} + k^{\mu\rho}k_{\rho}^{\nu}g^{\alpha\beta} - kk^{\alpha\beta}g^{\mu\nu} - kk^{\mu\nu}g^{\alpha\beta} + kk^{\mu\alpha}g^{\nu\beta} + kk^{\nu\beta}g^{\mu\alpha} - 4P^{\mu\alpha\nu\beta},\end{aligned}\quad (\text{A13})$$

$$\begin{aligned}\overline{\mathcal{M}}_{\mu\nu} &:= -\frac{3}{4}k_{\mu}^{\alpha}k_{\nu}^{\beta}k_{\alpha}^{\gamma}k_{\beta}^{\gamma} + \frac{3}{8}k_{\rho}^{\sigma}k_{\sigma}^{\rho}k_{\mu}^{\alpha}k_{\nu\alpha} + \frac{1}{4}k_{\rho}^{\sigma}k_{\lambda}^{\rho}k_{\sigma}^{\lambda}k_{\mu\nu} + \frac{5}{8}kk_{\mu}^{\alpha}k_{\nu}^{\beta}k_{\alpha\beta} - \frac{5}{16}kk_{\rho}^{\sigma}k_{\sigma}^{\rho}k_{\mu\nu} \\ &\quad - \frac{1}{4}k^2k_{\mu}^{\alpha}k_{\nu\alpha} + \frac{1}{16}k^2k_{\mu\nu} + \frac{1}{2}(R_{\mu\alpha\beta\gamma}k_{\nu}^{\gamma}k^{\alpha\beta} + R_{\nu\alpha\beta\gamma}k_{\mu}^{\gamma}k^{\alpha\beta}) - \frac{1}{2}R_{\alpha\beta}k_{\mu}^{\alpha}k_{\nu}^{\beta} + \frac{1}{2}R_{\alpha\beta}k^{\alpha\beta}k_{\mu\nu} \\ &\quad - \frac{1}{4}R_{\mu\alpha\nu\beta}k^{\alpha\gamma}k_{\gamma}^{\beta} + \frac{1}{2}kR_{\mu\alpha\nu\beta}k^{\alpha\beta} - k^{\alpha\beta}(R_{\mu\alpha}k_{\nu\beta} + R_{\nu\alpha}k_{\mu\beta}) + \frac{1}{2}k(R_{\mu\alpha}k_{\nu}^{\alpha} + R_{\nu\alpha}k_{\mu}^{\alpha}) \\ &\quad + \frac{3}{4}R_{\mu\nu}(3k_{\rho}^{\sigma}k_{\sigma}^{\rho} - k^2) + \frac{1}{2}R(2k_{\mu}^{\alpha}k_{\nu}^{\alpha} - kk_{\mu\nu}),\end{aligned}\quad (\text{A14})$$

$$\begin{aligned}\overline{\mathcal{L}} &:= -\frac{7}{8}k_{\beta}^{\alpha}k_{\gamma}^{\beta}k_{\lambda}^{\gamma}k_{\alpha}^{\lambda} + \frac{7}{16}(k_{\rho}^{\sigma}k_{\sigma}^{\rho})^2 + kk_{\beta}^{\alpha}k_{\gamma}^{\beta}k_{\alpha}^{\gamma} - \frac{5}{8}k^2(k_{\rho}^{\sigma}k_{\sigma}^{\rho}) + \frac{1}{16}k^4 \\ &\quad + R_{\alpha\beta\gamma\delta}k^{\alpha\delta}k^{\beta\gamma} - 4R_{\alpha\beta}k^{\alpha\gamma}k_{\gamma}^{\beta} + 2kk^{\alpha\beta}R_{\alpha\beta} + \frac{3}{2}Rk_{\rho}^{\sigma}k_{\sigma}^{\rho} - \frac{1}{2}Rk^2.\end{aligned}\quad (\text{A15})$$

There are no quadratic terms such as $R_{\mu\alpha}R_{\nu}^{\alpha}$ in $\mathcal{H}_{\mu\nu}$

as the Gauss-Bonnet tensor is identically zero in four

dimensions. As is well-known [3], neither \mathcal{H}_{yy} nor $\mathcal{H}_{y\mu}$ contain $l_{\mu\nu}$; as for $\mathcal{H}_{\mu\nu}$ it is quasi-linear in $l_{\mu\nu}$.

The Gauss-Bonnet field equations at lowest order in y can be seen as a boundary condition problem: given a 4-dimensional metric $g_{\mu\nu}$ and an extrinsic curvature $k_{\mu\nu}/2$ satisfying the constraints [the (yy) and $(y\mu)$ components of the field equations], then the $(\mu\nu)$ component gives $l_{\mu\nu}$ which is the metric in the bulk at order $\mathcal{O}(y^2)$. When $\alpha \neq 0$, $l_{\mu\nu}$ is only known implicitly because of the quasi-linearity of the Gauss-Bonnet tensor \mathcal{H}_{AB} .

Now the junction conditions relate the extrinsic curvature to the energy-momentum tensor on the brane [12, 13]. They can in fact be read off from the $(y\mu)$ component of the field equations, and are written in terms of $k_{\mu\nu}$ and $j_{\mu\nu}$ as

$$T^\mu_\nu = \sigma \delta^\mu_\nu - k^\mu_\nu + \delta^\mu_\nu k - \alpha \left(\frac{3}{2} j^\mu_\nu - \frac{1}{2} \delta^\mu_\nu j - 4 P^\mu_{\rho\nu\sigma} k^{\rho\sigma} \right), \quad (\text{A16})$$

where the constant $\sigma := (6/\ell) [1 - (4\alpha/3\ell^2)]$ is such that $k_{\mu\nu} = -(2/\ell)g_{\mu\nu}$, together with $R_{\mu\nu\rho\sigma} = 0$, solves the brane equations (A4), ℓ being related to Λ by $\Lambda = -(6/\ell^2) (1 - 2\alpha/\ell^2)$. (Here and hereafter we set $\kappa^2 = 1$.) The $(y\mu)$ component of the brane field equations is then equivalent to the conservation of the energy-momentum

tensor: $D_\nu T^\nu_\mu = 0$.

In Einstein gravity ($\alpha = 0$), inverting the junction equations to express $k_{\mu\nu}$ in terms of $T_{\mu\nu}$ is elementary: $k_{\mu\nu} = -(2/\ell)g_{\mu\nu} - [T_{\mu\nu} - (1/3)Tg_{\mu\nu}]$. Hence the brane field equations (A4) can readily be written in terms of $T_{\mu\nu}$ rather than $k_{\mu\nu}$, and we arrive at the effective equations on the brane [9]

$$G_{\mu\nu} = \kappa_4^2 T_{\mu\nu} - E_{\mu\nu} - \frac{1}{4} T_{\mu\rho} T^\rho_\nu + \frac{1}{12} T T_{\mu\nu} + \frac{1}{8} g_{\mu\nu} T^\rho_\sigma T^\sigma_\rho - \frac{1}{24} T^2 g_{\mu\nu}, \quad (\text{A17})$$

with the identification $\kappa_4^2 = 1/\ell$. In deriving the above equation we introduced the electric part of the Weyl tensor and rewrote $l_{\mu\nu}$ as

$$l_{\mu\nu} = -2E_{\mu\nu} - \frac{1}{3} \Lambda g_{\mu\nu} + \frac{1}{2} k_{\mu\rho} k^\rho_\nu. \quad (\text{A18})$$

In Einstein-Gauss-Bonnet gravity, the junction equations (A16) cannot explicitly be inverted to give $k_{\mu\nu}$ in terms of $T_{\mu\nu}$. Hence, the gradient expansion scheme developed in the main text, which consists in writing $k_{\mu\nu} = -(2/\ell)g_{\mu\nu} + \epsilon_{\mu\nu}$ and expanding (A16) to second order in $\epsilon_{\mu\nu}$, is a way to solve iteratively the brane equations in a closed form.

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